On Average Properties of Inhomogeneous Fluids in General Relativity: Perfect Fluid Cosmologies

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Summary: For general relativistic spacetimes filled with an irrotational perfect fluid a generalized form of Friedmann's equations governing the expansion factor of spatially averaged portions of inhomogeneous cosmologies is derived. The averaging problem for scalar quantities is condensed into the problem of finding an 'effective equation of state' including kinematical as well as dynamical 'backreaction' terms that measure the departure from a standard FLRW cosmology. Applications of the averaged models are outlined including radiation–dominated and scalar field cosmologies (inflationary and dilaton/string cosmologies). In particular, the averaged equations show that the averaged scalar curvature must generically change in the course of structure formation, that an averaged inhomogeneous radiation cosmos does not follow the evolution of the standard homogeneous–isotropic model, and that an averaged inhomogeneous perfect fluid features kinematical 'backreaction' terms that, in some cases, act like a free scalar field source. The free scalar field (dilaton) itself, modelled by a 'stiff' fluid, is singled out as a special inhomogeneous case where the averaged equations assume a simple form.

1. Introduction

The present paper continues a line of research on average properties of inhomogeneous fluids in general relativity that is based on a simple and intuitive averaging procedure. The simplification is guided by the restriction to scalar dynamical variables and to standard volume integration, in which case averaging is straightforward (for a discussion of alternative procedures see, e.g., Stoeger et al. 1999). Averaging is aimed at the construction of an *effective dynamics* of spatial portions of the Universe from which, in principle, observable average characteristics can be inferred like Hubble's constant, the effective 3–Ricci scalar curvature and the mean density of a given spatial domain, which is bounded by the limits of observation. Naturally, this view entails a scale–dependent description of inhomogeneous cosmologies. In the case where the extension of the (simply–connected) spatial domain to the whole Universe is possible, such a description may allow to draw conclusions about global properties of the world models.

Paper I (Buchert 2000) was concerned with 'dust cosmologies' restricting attention to the most popular inhomogeneous cosmologies. It is, however, desirable to extend the regime of application of an 'effective' (i.e. averaged) dynamics to a wider range of spatial and temporal scales than that covered by the matter model 'dust'. This is the motivation of the present work which presents the results for a large class of perfect fluid cosmologies.

This class opens quite a piece of new terrain: it covers radiation–dominated cosmologies, scalar field cosmologies including inhomogeneous dilaton/string cosmologies and inflationary cosmologies. It also extends the range of validity concerning averages of large–scale structure formation models for collisionless matter, in which case the presence of a pressure–force that counteracts gravity is implied by the development of multi–streaming within high–density regions; here, it provides a phenomenological extension by including physics on smaller spatial scales for the evolution of structure (see: Buchert & Domínguez 1998, Buchert et al. 1999 in Newtonian cosmology; Maartens et al. 1999 in GR).

This paper is organized as follows. Sect. 2 presents Einstein's equations for irrotational perfect fluids with the choice of foliation into flow–orthogonal hypersurfaces. Averaging the scalar parts of Einstein's equations is investigated in Sect. 3. The result is presented in a *Theorem* in Subsect. 3.2., which shows that the average expansion of inhomogeneous models is controlled by 'kinematical backreaction' due to shear and expansion fluctuations, and by 'dynamical backreaction' due to a non–vanishing pressure gradient in the hypersurfaces. It is manifest that the simple relation between averaged 3–Ricci scalar curvature and 'kinematical backreaction', as found for the matter model 'dust' in Paper I (Buchert 2000), is supplemented by several additional effects: besides dynamical contributions to 'backreaction' the averaged energy– and momentum conservation laws do not yield conservation laws for the averaged fields. This leads to more drastic changes in the average flow compared with the standard model. *Corollary 1* and *Corollary 2* present compact formulations of the averaged equations for *effective* sources. The relations between additional sources in the generalized Friedmann equations are so formally reduced to the search for an *effective equation of state*. While Sect. 3 appears rather formal, especially because the presence of pressure involves an inhomogeneous lapse function and so impairs the simplicity of the equations, emphasis is focused on the application side in Sect. 4. There we discuss some relevant subcases that are members of the family of barotropic fluids: averaged 'dust' models are recovered from the more general framework, averaged radiation–dominated models display deviations from a standard radiation cosmos, even if 'kinematical backreaction' is absent, and the application to inhomogeneous scalar field cosmologies is outlined.

2. Einstein's Equations for Perfect Fluids

2.1. Choice of Foliation and Dynamical Variables

We shall assume for the cosmic fluid that it is perfect and irrotational, so that we can introduce a foliation of spacetime into hypersurfaces orthogonal to the 4–velocity. It is not a problem to allow for a 'tilted' slicing in order to include, e.g., vorticity (see, e.g., MacCallum & Taub (1972), King & Ellis (1973), Hwang & Vishniac (1990), and Ellis et al. (1990)). For the applications we have in mind and also to keep the present investigation transparent, we shall evaluate everything for this class of fluids.

For the purpose of averaging we shall consider a compact and simply–connected domain contained within spatial hypersurfaces that are specified below. This domain will be followed along the flow lines of the fluid elements; thus we require that the total restmass of the fluid within the domain be conserved.

Let us first consider the (conserved) restmass flux vector[1]

$$M^\mu := \varrho u^\mu \quad ; \quad M^\mu_{\ ;\mu} = 0 \quad ; \quad \varrho > 0 \quad , \tag{1a}$$

where ϱ is the restmass density and the flow lines are integral curves of the 4–velocity u^μ. Confining ourselves to irrotational fluids guarantees the existence of a scalar function S, such that

$$u^\mu =: \frac{-\partial^\mu S}{h} \quad , \tag{1b}$$

where the function h will be identified below. It normalizes the 4–gradient $\partial^\mu S$ so that $u^\mu u_\mu = -1$,

$$h = \sqrt{-\partial^\alpha S \partial_\alpha S} = u^\mu \partial_\mu S =: \dot{S} \ > \ 0 \quad . \tag{1c}$$

The overdot stands for the material derivative operator along the flow lines of any tensor field $\mathcal{F}$ as defined covariantly by

$$\dot{\mathcal{F}} := u^\mu \mathcal{F}_{\ ;\mu} \quad . \tag{1d}$$

We shall aim at a covariant description of the fluid flow with respect to the natural foliation of spacetime into hypersurfaces $S = const.$ representing the 3–dimensional 'wave fronts' (for the covariant fluid approach compare Ellis & Bruni 1989, Bruni et al. 1990a,b, Dunsby et al. 1992). With our choice of the fluid's 4–velocity (1b) we have to assure that it remains time–like and, hence, the hypersurfaces $S = const.$ space–like. For this to be true the 4–gradient of the scalar field has to be time–like,

$$\partial_\alpha S \partial^\alpha S = -h^2 < 0 \quad . \tag{1g}$$

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[1]Greek indices run through 0...3, while latin indices run through 1...3; summation over repeated indices is understood. A semicolon will denote covariant derivative with respect to the 4–metric with signature $(-,+,+,+)$; the units are such that $c = 1$.

(For $h \in \mathbb{R}$ this is always true.) As already noted, the definition (1b) implies that u^μ is irrotational (2a); it also implies that the covariant spatial gradient of S in the hypersurfaces of constant S, denoted by $S_{||\mu}$, vanishes (2b),

$$\omega_{\mu\nu} = h_\mu^{\ \alpha} h_\nu^{\ \beta} u_{[\alpha;\beta]} = -h_\mu^{\ \alpha} h_\nu^{\ \beta} (\frac{1}{h}\partial_{[\alpha} S)_{;\beta]} = 0 \ \ ; \tag{2a}$$

$$S_{||\mu} = h_\mu^{\ \nu} \partial_\nu S = \partial_\mu S + u_\mu \dot{S} \ = \ 0 \ \ , \tag{2b}$$

where $h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor into the hypersurfaces $S = const.$ orthogonal to the integral curves of the 4–velocity u^μ, $h_{\mu\nu}u^\nu = 0$.

Hence, $S(t)$ and $h(X^i, t)$ play the role of 'phase' and 'amplitude' of the fluid's 'wave fronts', respectively.

On these hypersurfaces we introduce the 3–metric g_{ij} (the first fundamental form) that is induced by the projection, as well as the extrinsic curvature tensor (the second fundamental form):

$$h_{ij} := g_{\mu\nu} h^\mu_{\ i} h^\nu_{\ j} = g_{ij} \quad ; \quad K_{ij} := -u_{\mu;\nu} h^\mu_{\ i} h^\nu_{\ j} = -u_{i;j} \ \ . \tag{3a, b}$$

The final result will be covariant with respect to the given foliation, but we shall label the flow lines by introducing (intrinsic) Gaussian coordinates X^i that appear in the line–element

$$ds^2 = -N^2 dt^2 + g_{ij} dX^i dX^j \ \ . \tag{4a}$$

Since by this choice of coordinates the velocities in 3–space vanish we are entitled to call X^i Lagrangian coordinates. In the language of the ADM formalism, which is put into perspective in the Appendix, we have a vanishing shift vector in the hypersurfaces and the lapse function (together with the 3–metric) encodes the inhomogeneities. For scalar functions $\mathcal{F} = \psi$ the covariant derivative (1d) reduces to the total (or Lagrangian) derivative along the flow lines,

$$\frac{d}{d\tau}\psi := \frac{dx^\mu}{dt}\partial_\mu \psi = u^\mu \partial_\mu \psi = \frac{1}{N}\partial_t \psi \ \ , \tag{4b}$$

where N is the (inhomogeneous) lapse function. It is crucial to note that the latter operator corresponds to a total time derivative with respect to *proper time* τ, which can be defined by $\tau := \int N dt$.

For later discussions we may express the symmetric tensor K_{ij}, or the expansion tensor $\Theta_{ij} := -K_{ij}$, respectively, in terms of kinematical quantities and their scalar invariants (Ehlers 1961). We decompose Θ_{ij} into its trace–free symmetric 'shear tensor' $\sigma_{ij} := \sigma_{\mu\nu} h^\mu_{\ i} h^\nu_{\ j}$, $\sigma_{\mu\nu}u^\nu = 0$, and its trace, the 'rate of expansion' $\theta := u^\alpha_{\ ;\alpha}$. From the decomposition $u_{\mu;\nu} = \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} - \dot{u}_\mu u_\nu$ we have:

$$-K_{ij} = \Theta_{ij} = \sigma_{ij} + \frac{1}{3}\theta g_{ij} \quad ; \quad -K = \theta \ \ . \tag{5a}$$

The tensor has three principal scalar invariants; in what follows we shall use two of them:

$$\mathbf{I} := -K = \theta \quad ; \quad 2\mathbf{II} := K^2 - K^i_{\ j} K^j_{\ i} = \frac{2}{3}\theta^2 - 2\sigma^2 \ \ , \tag{5b, c}$$

where we have introduced the 'rate of shear' σ by $\sigma^2 := \frac{1}{2}\sigma^i{}_j\sigma^j{}_i$.

2.2. Basic Equations in 3 + 1 Form

Einstein's equations for an irrotational perfect fluid with the energy–momentum tensor

$$T_{\mu\nu} := \varepsilon u_\mu u_\nu + p h_{\mu\nu} \quad , \tag{6}$$

with energy density ε and pressure p, may be cast into a set of 'constraint equations', the Hamiltonian and momentum constraints[2],

$$\mathcal{R} + K^2 - K^i{}_j K^j{}_i = 16\pi G\varepsilon \quad , \tag{7a}$$

$$K^i{}_{j||i} - K_{|j} = 0 \quad , \tag{7b}$$

and 'evolution equations' for the the two fundamental forms:

$$\frac{d}{d\tau} g_{ij} = -2 \; g_{ik} K^k{}_j \quad , \tag{7c}$$

$$\frac{d}{d\tau} K^i{}_j = K K^i{}_j + \mathcal{R}^i{}_j - 4\pi G \delta^i{}_j \left(\epsilon - p\right) - \left(a^i{}_{||j} + a^i a_j\right) \quad , \tag{7d}$$

where the acceleration is completely contained in the hypersurfaces of constant S and is defined as

$$a_i = h^\mu{}_i a_\mu \quad , \quad a^\mu := u^\nu u^\mu{}_{;\nu} = \dot{u}^\mu \quad , \quad a^\mu u_\mu = 0 \quad , \tag{8a}$$

and $\mathcal{R} := \mathcal{R}^i{}_i$, $K := K^i{}_i$ denote the traces of the spatial Ricci tensor $\mathcal{R}^i{}_j$ and the extrinsic curvature tensor, respectively. Below, we shall only average the 4−divergence $\mathcal{A}$ of the acceleration field:

$$\mathcal{A} := a^\mu{}_{;\mu} = a^i{}_{||i} + a^i a_i \quad . \tag{8b}$$

A nonvanishing acceleration is a consequence of the fact that the pressure term forces deviations from a geodesic flow. From $T^{\mu\nu}{}_{;\nu} = 0$ we derive the energy and momentum conservation laws:

$$u_\mu T^{\mu\nu}{}_{;\nu} = 0 \quad \Leftrightarrow \quad \dot{\varepsilon} = -\theta(\varepsilon + p) \quad , \tag{9a}$$

$$h_{\mu\alpha} T^{\mu\nu}{}_{;\nu} = 0 \quad \Leftrightarrow \quad \dot{u}_\alpha = a_\alpha = -\frac{1}{\varepsilon + p} \partial_\mu p h^\mu{}_\alpha = -\frac{1}{\varepsilon + p} p_{|\alpha} \quad . \tag{9b}$$

Hence,

$$a_i = -\frac{1}{\varepsilon + p} p_{|i} \quad , \tag{9c}$$

[2] As before, a double vertical slash abbreviates the covariant derivative with respect to the 3–metric g_{ij}; for scalars it reduces to the partial derivative with respect to Lagrangian coordinates denoted by a single vertical slash.

and

$$\mathcal{A} = -\frac{1}{\varepsilon + p} p^{|i}{}_{||i} + \frac{2}{(\varepsilon + p)^2} p^{|i} p_{|i} + \frac{1}{(\varepsilon + p)^2} p^{|i} \varepsilon_{|i} \quad . \tag{9d}$$

From Eq. (1a) we also have the continuity equation for the restmass density

$$\dot{\varrho} + \theta \varrho = 0 \quad . \tag{9e}$$

According to (1c), $\dot{S} = \frac{1}{N}\partial_t S(t) = h$, we can write Eq. (9d) completely in terms of the magnitude h and its spatial derivatives:

$$\mathcal{A} = \left(\frac{N^{|i}}{N}\right)_{||i} = -\frac{1}{h} h^{|i}{}_{||i} + \frac{2}{h^2} h^{|i} h_{|i} \; = \; h \left(\frac{1}{h}\right)^{|i}{}_{||i} \quad . \tag{10a}$$

Two other derived formulas will be used in what follows. First, Raychaudhuri's equation, which follows by taking the trace of (7d) and inserting (7a):

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi G \left(\varepsilon + 3p\right) + \mathcal{A} \quad , \tag{10b}$$

and, second, an expression for the spatial Ricci curvature scalar in terms of the energy source terms, the restmass density, the magnitude h and its spatial derivatives: eliminating $2\mathbf{II} = \frac{2}{3}\theta^2 - 2\sigma^2$ from Eq. (10b) and, using Eq. (5c), from the Hamiltonian constraint Eq. (7a), we obtain with Eq. (9e):

$$\mathcal{R} = 12\pi G(\varepsilon - p) - \varrho \frac{d^2}{d\tau^2}\left(\frac{1}{\varrho}\right) + h \left(\frac{1}{h}\right)^{|i}{}_{||i} \quad . \tag{10c}$$

2.3. Thermodynamics of the Fluid

First we note that the energy and restmass conservation laws Eqs. (9a,e) are equivalent according to the first law of thermodynamics,

$$\frac{d\varrho}{\varrho} = \frac{d\varepsilon}{\varepsilon + p} =: \frac{ds}{s} \quad , \tag{11a}$$

upon dividing by $d\tau$. The latter equality defines the entropy density s that obeys the conservation law

$$(s u^{\mu})_{;\mu} = 0 \;\; ; \;\; \dot{s} + \theta s = 0 \quad . \tag{11b}$$

For closing the system of Einstein equations we need to identify a concrete matter model. Specific models are obtained by invoking a local 'equation of state' relating the pressure with the other dynamical variables. We shall discuss examples that are all members of the class of 'barotropic fluids', i.e., $p=\alpha(\varepsilon)$ is assumed to be locally given and, in particular, the function α is the same for each fluid element (at each trajectory). The special inhomogeneous fluid cases discussed will all be contained in the simpler class $\alpha(\varepsilon)=\gamma\varepsilon$ with $\gamma=const.$, a 'dust' matter model ($\gamma=0$), a 'radiation fluid' ($\gamma=\frac{1}{3}$), and a 'stiff fluid' corresponding to a free minimally coupled scalar field ($\gamma=1$).

We shall now identify the normalization amplitude h. Let us first derive h for a 'barotropic fluid'. The momentum conservation law implies

$$\frac{N_{|i}}{N}=-\frac{h_{|i}}{h}=-\frac{p_{|i}}{\varepsilon+p} \quad . \tag{11b}$$

Defining $\Pi:=\int\frac{dp}{\varepsilon+p}$, with $\varepsilon=\alpha^{-1}(p)$, we may write Eq. (11b) as $(h_0\ln(\frac{h}{h_0})+\Pi)_{|i}=0$, which may be integrated to give

$$h\propto\exp\Pi \quad , \tag{11c}$$

up to a time–dependent integration function. Hence, we may write

$$\frac{dh}{h}=\frac{dp}{\varepsilon+p} \quad . \tag{11d}$$

In general, we identify the magnitude h with the 'injection energy per fluid element and unit restmass' (Israel 1976),

$$h:=\frac{\varepsilon+p}{\varrho} \quad , \tag{11e}$$

which is related to the relativistic enthalpy $\eta:=\frac{\varepsilon+p}{n}$ by $h=\eta/m$ with m the unit restmass of a fluid element, and n the baryon density. Note that Eq. (11d) holds by defining h as in Eq. (11e) as a result of the conservation laws Eqs. (9a,e), since from Eq. (11a),

$$d\varepsilon=hd\varrho \quad . \tag{11f}$$

For a barotropic fluid we can easily see that ε is a function of the restmass density only and, hence, h is a function of ϱ. The evolution equation for h in this case (and in the simpler case $p=\gamma\varepsilon$) reads:

$$\dot{h}+\theta\alpha'(\varepsilon)h=\dot{h}+\gamma\theta h=0 \quad ; \tag{11g}$$

h obeys a simple continuity equation in the case of a 'stiff' fluid with $\gamma=1$.

The discussion of special cases will be resumed in Sect. 4.

3. The Averaged System

3.1. The Averaging Procedure

Spatially averaging equations for scalar fields is a covariant operation given a foliation of spacetime. Therefore, we shall in what follows only consider scalar functions $\Upsilon(X^i,t)$. We shall define the averaging operation by the usual spatial volume average performed on an arbitrary compact support of the fluid $\mathcal{D}$ contained within the hypersurfaces $S(t) = const.$:

$$\langle\Upsilon\rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} \Upsilon \; J d^3X \quad , \quad J := \sqrt{\det(g_{ij})} \quad . \tag{12a}$$

The volume of the region itself (set $\Upsilon = 1$) is given by $V_{\mathcal{D}}(t) := \int_{\mathcal{D}} J d^3X$.
We also introduce a dimensionless scale factor via the volume (normalized by the volume of the initial domain $V_{\mathcal{D}o}$):

$$a_{\mathcal{D}}(t) := \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}o}}\right)^{1/3} \quad . \tag{12b}$$

This means that we are only interested in the effective dynamics of the domain; $a_{\mathcal{D}}$ will be a functional of the domain's shape (dictated by the metric) and position. Since the domains follow the flow lines, the total restmass $M_{\mathcal{D}} := \int_{\mathcal{D}} \varrho J d^3X$ contained in a given domain is conserved.
The following formulas are crucial for evaluating averages. Taking the trace of Eq. (7c), written in the form

$$K^i_{\ j} = -\frac{1}{2}g^{ik}\frac{d}{d\tau}g_{kj} \quad ,$$

we obtain with $\frac{1}{2}g^{ik}\frac{d}{d\tau}g_{ki} = (\ln J)^{\bullet}$ the identity

$$\dot{J} = \theta J \quad . \tag{12c}$$

The rate of change of the volume $V(t)$ in the hypersurfaces $S(t) = const.$ is evaluated by taking the partial time derivative of the volume and dividing by the volume. Since ∂_t and d^3X commute (but not $\frac{d}{d\tau}$ and d^3X !) we obtain:

$$\frac{\partial_t V_{\mathcal{D}}(t)}{V_{\mathcal{D}}(t)} = \frac{1}{V_{\mathcal{D}}(t)}\int_{\mathcal{D}} \partial_t J d^3X = \frac{1}{V_{\mathcal{D}}(t)}\int_{\mathcal{D}} N\dot{J} d^3X = \frac{1}{V_{\mathcal{D}}(t)}\int_{\mathcal{D}} N\theta J d^3X = \langle N\theta\rangle_{\mathcal{D}} \quad . \tag{12d}$$

Introducing the scaled (t–)expansion $\tilde{\theta} := N\theta$ we define an effective (t–)Hubble function in the hypersurfaces by

$$\langle\tilde{\theta}\rangle_{\mathcal{D}} = \frac{\partial_t V_{\mathcal{D}}(t)}{V_{\mathcal{D}}(t)} = 3\frac{\partial_t a_{\mathcal{D}}}{a_{\mathcal{D}}} =: 3\tilde{H}_{\mathcal{D}} \quad . \tag{12e}$$

(Notice that we reserve the overdot for the covariant derivative.)

It is now straightforward to prove the following *Lemma* for an arbitrary scalar field $\Upsilon(X^i, t)$:

Lemma (*Commutation rule*)

$$\partial_t \langle \Upsilon \rangle_{\mathcal{D}} - \langle \partial_t \Upsilon \rangle_{\mathcal{D}} = \langle \Upsilon \tilde{\theta} \rangle_{\mathcal{D}} - \langle \Upsilon \rangle_{\mathcal{D}} \langle \tilde{\theta} \rangle_{\mathcal{D}} \quad , \tag{12f}$$

or, alternatively,

$$\partial_t \langle \Upsilon \rangle_{\mathcal{D}} + 3 \tilde{H}_{\mathcal{D}} \langle \Upsilon \rangle_{\mathcal{D}} = \langle \partial_t \Upsilon + \Upsilon \tilde{\theta} \rangle_{\mathcal{D}} \quad . \tag{12g}$$

A simple application of this *Lemma* is the proof that the total restmass in a domain is conserved: let $\Upsilon = \varrho$, then $\partial_t \langle \varrho \rangle_{\mathcal{D}} + 3 \tilde{H}_{\mathcal{D}} \langle \varrho \rangle_{\mathcal{D}} = \langle \partial_t \varrho + \varrho \tilde{\theta} \rangle_{\mathcal{D}} = 0$ according to the local conservation law Eq. (9e). **q.e.d.**

3.2. Averaged Equations for Irrotational Perfect Fluids

Averaging Raychaudhuri's equation (10b) and the Hamiltonian constraint (7a) with the help of the prescribed procedure, we end up with the following two equations for the scale factor of the averaging domain which we may formulate in the form of a theorem:

Theorem – Part I (*Equations for the effective scale factor*)

The spatially averaged equations for the scale factor $a_{\mathcal{D}}$, respecting restmass conservation, read:
averaged Raychaudhuri equation for the scaled (t–)densities $\tilde{\varepsilon} := N^2 \varepsilon$ and $\tilde{p} := N^2 p$:

$$3 \frac{\partial_t^2 a_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \langle \tilde{\varepsilon} + 3 \tilde{p} \rangle_{\mathcal{D}} \;=\; \tilde{\mathcal{Q}}_{\mathcal{D}} + \tilde{\mathcal{P}}_{\mathcal{D}} \quad ; \tag{13a}$$

averaged Hamiltonian constraint:

$$6 \tilde{H}_{\mathcal{D}}^2 - 16\pi G \langle \tilde{\varepsilon} \rangle_{\mathcal{D}} \;=\; - \Big(\tilde{\mathcal{Q}}_{\mathcal{D}} + \langle \tilde{\mathcal{R}} \rangle_{\mathcal{D}} \Big) \quad , \tag{13b}$$

where we have introduced the scaled spatial (t–)Ricci scalar $\tilde{\mathcal{R}} := N^2 \mathcal{R}$, and we have separated off the domain dependent 'backreaction' terms: the *kinematical backreaction*,

$$\tilde{\mathcal{Q}}_{\mathcal{D}} := [2 \langle N^2 \mathbf{II} \rangle_{\mathcal{D}} - \frac{2}{3} \langle N \mathbf{I} \rangle_{\mathcal{D}}^2] \;=\; \frac{2}{3} \langle \Big(\tilde{\theta} - \langle \tilde{\theta} \rangle_{\mathcal{D}} \Big)^2 \rangle_{\mathcal{D}} - 2 \langle \tilde{\sigma}^2 \rangle_{\mathcal{D}} \quad , \tag{13c}$$

with the scaled (t–)shear scalar $\tilde{\sigma} := N\sigma$, and the *dynamical backreaction*,

$$\tilde{\mathcal{P}}_{\mathcal{D}} := \langle \tilde{\mathcal{A}} \rangle_{\mathcal{D}} + \langle \dot{N} \tilde{\theta} \rangle_{\mathcal{D}} \quad , \tag{13d}$$

with the scaled (t–)acceleration divergence $\tilde{\mathcal{A}} := N^2 \mathcal{A}$.

Note: Eq. (13a) can also be obtained by an argument given by Yodzis (1974), which is summarized in Appendix C of (Paper I).

The source terms on the r.–h.–s. of Eq. (13a) include the 'kinematical backreaction' (13c) that describes the impact of inhomogeneities on the scale factor due to averaged shear and expansion fluctuations. It vanishes for the standard FLRW cosmologies. Additionally, Eq. (13a) features another 'dynamical backreaction' term $\langle\tilde{\mathcal{A}}\rangle_{\mathcal{D}}$ together with a technical term due to the change of the lapse function. The former term also vanishes for standard FLRW cosmologies; both terms vanish for zero pressure. Note that the averaged Hamiltonian constraint does not involve pressure terms as expected.

These equations show that the averaged shear fluctuations tend to increase the expansion rate similar to the effect of the averaged energy source terms (provided the energy condition $\langle\tilde{\varepsilon}+3\tilde{p}\rangle_{\mathcal{D}}>0$ holds), while the averaged expansion fluctuations work in the direction of stabilizing structures. Pressure forces can do both; the sign of the averaged divergence of the 4–acceleration can be positive or negative. In the Newtonian framework one can show that, to a first–order approximation, the combined effect of gravity and pressure leads to stabilization of structures (Buchert & Domínguez 1998, Buchert et al. 1999, Adler & Buchert 1999). Note, however, that since pressure is a source of the gravitational field energy too, it is harder to oppose the gravitational collapse than in the corresponding Newtonian treatment (compare the terms which add positive contributions in Eq. (9d) with their Newtonian analogues).

We proceed by calculating the integrability condition for the system of equations (13a,b), i.e., we shall answer the question which equation has to hold in order that (13b) be the integral of (13a). For this end we take the partial time–derivative of (13b) and insert into the result again our starting set of equations (13a) and (13b). We get:

Theorem – Part II (*Integrability and energy balance conditions*)

Eq. (13b) is an integral of Eq. (13a), iff

$$\partial_t\tilde{\mathcal{Q}}_{\mathcal{D}}+6\tilde{H}_{\mathcal{D}}\tilde{\mathcal{Q}}_{\mathcal{D}}+\partial_t\langle\tilde{\mathcal{R}}\rangle_{\mathcal{D}}+2\tilde{H}_{\mathcal{D}}\langle\tilde{\mathcal{R}}\rangle_{\mathcal{D}}+4\tilde{H}_{\mathcal{D}}\tilde{\mathcal{P}}_{\mathcal{D}}$$
$$-16\pi G[\partial_t\langle\tilde{\varepsilon}\rangle_{\mathcal{D}}+3\tilde{H}_{\mathcal{D}}\langle\tilde{\varepsilon}+\tilde{p}\rangle_{\mathcal{D}}]\;=\;0\quad . \qquad (14a)$$

The expression involving the energy density and the pressure does not vanish in general. To see this we average the local energy conservation law (9a). We obtain:

$$\partial_t\langle\varepsilon\rangle_{\mathcal{D}}+3\tilde{H}_{\mathcal{D}}\langle\varepsilon+p\rangle_{\mathcal{D}}=\langle\partial_t p\rangle_{\mathcal{D}}-\partial_t\langle p\rangle_{\mathcal{D}}\quad . \qquad (14b)$$

For the scaled (t–)variables we accordingly have for the local law:

$$\partial_t\tilde{\varepsilon}+\tilde{\theta}(\tilde{\varepsilon}+\tilde{p})=2\dot{N}\tilde{\varepsilon}\quad , \qquad (14c)$$

and for the average:

$$\partial_t\langle\tilde{\varepsilon}\rangle_{\mathcal{D}}+3\tilde{H}_{\mathcal{D}}\langle\tilde{\varepsilon}+\tilde{p}\rangle_{\mathcal{D}}=\langle\partial_t\tilde{p}\rangle_{\mathcal{D}}-\partial_t\langle\tilde{p}\rangle_{\mathcal{D}}+\langle 2\dot{N}\tilde{\varepsilon}\rangle_{\mathcal{D}}\quad . \qquad (14d)$$

This shows that the pressure term introduces a possibly interesting effect. In the 'dust' case (Paper I: Corollary 1) the averaged fields obey the same equations as the local fields provided we use their representation in terms of invariants of the second fundamental form. Here, this is no longer true. In particular, Eq. (14b) shows that the averaged energy conservation law invokes non–commuting terms that are nonzero for inhomogeneous fluids. Thus, even if both the 'kinematical' and 'dynamical backreaction' terms are assumed to be negligible or cancel for "some" reason, the averaged model is different from the standard homogeneous–isotropic models. This fact will be manifest in the following more compact alternative representations of the averaged equations.

Corollary 1 (*Averaged equations: first effective form*)

Let us define effective densities as follows:

$$\varepsilon^{(1)}_{\rm eff} := \langle\tilde\varepsilon\rangle_{\mathcal D} - \frac{\tilde{\mathcal Q}_{\mathcal D}}{16\pi G} \quad , \tag{15a}$$

$$p^{(1)}_{\rm eff} := \langle\tilde p\rangle_{\mathcal D} - \frac{\tilde{\mathcal Q}_{\mathcal D}}{16\pi G} - \frac{\tilde{\mathcal P}_{\mathcal D}}{12\pi G} \quad . \tag{15b}$$

Then, the averaged equations can be cast into a form similar to the standard Friedmann equations:

$$3\frac{\partial_t^2 a_{\mathcal D}}{a_{\mathcal D}} + 4\pi G\left(\varepsilon^{(1)}_{\rm eff} + 3p^{(1)}_{\rm eff}\right) \;=\; 0 \quad ; \tag{15c}$$

$$6\tilde H^2_{\mathcal D} + \langle\tilde{\mathcal R}\rangle_{\mathcal D} - 16\pi G\varepsilon^{(1)}_{\rm eff} \;=\; 0 \quad , \tag{15d}$$

and the integrability condition of (15c) to yield (15d) has the form of a balance equation between the effective sources and the averaged spatial (t–)Ricci scalar:

$$\partial_t\varepsilon^{(1)}_{\rm eff} + 3\tilde H_{\mathcal D}\left(\varepsilon^{(1)}_{\rm eff} + p^{(1)}_{\rm eff}\right) = \frac{1}{16\pi G}\left(\partial_t\langle\tilde{\mathcal R}\rangle_{\mathcal D} + 2\tilde H_{\mathcal D}\langle\tilde{\mathcal R}\rangle_{\mathcal D}\right) \quad . \tag{15e}$$

The effective densities obey a conservation law, if the domains' curvature evolves like in a "small" FLRW cosmology, $\langle\tilde{\mathcal R}\rangle_{\mathcal D} = 0$, or $\langle\tilde{\mathcal R}\rangle_{\mathcal D} \propto a_{\mathcal D}^{-2}$, respectively. In particular, Eq. (15e) shows that in general the averaged densities are directly coupled to the evolution of the averaged spatial curvature.

Considering the averaged spatial t–Ricci scalar as an effective source as well, one may cast the equations into an even more elegant form.

Corollary 2 (*Averaged equations: second effective form*)

Defining

$$\varepsilon_{\rm eff}^{(2)} :=\langle\tilde\varepsilon\rangle_{\mathcal D} - \frac{\tilde{\mathcal Q}_{\mathcal D}}{16\pi G} - \frac{\langle\tilde{\mathcal R}\rangle_{\mathcal D}}{16\pi G} \quad , \tag{16a}$$

$$p_{\rm eff}^{(2)} :=\langle\tilde p\rangle_{\mathcal D} - \frac{\tilde{\mathcal Q}_{\mathcal D}}{16\pi G} + \frac{\langle\tilde{\mathcal R}\rangle_{\mathcal D}}{48\pi G} - \frac{\tilde{\mathcal P}_{\mathcal D}}{12\pi G} \quad , \tag{16b}$$

we obtain equations that assume the form of spatially 3–Ricci flat Friedmann cosmologies:

$$3\frac{\partial_t^2 a_{\mathcal D}}{a_{\mathcal D}} + 4\pi G\left(\varepsilon_{\rm eff}^{(2)} + 3p_{\rm eff}^{(2)}\right) \;=\; 0 \quad ; \tag{16c}$$

$$6\tilde H_{\mathcal D}^2 - 16\pi G\varepsilon_{\rm eff}^{(2)} \;=\; 0 \quad , \tag{16d}$$

and the integrability condition of (16c) to yield (16d) has exactly the form of a conservation law:

$$\partial_t\varepsilon_{\rm eff}^{(2)} + 3\tilde H_{\mathcal D}\left(\varepsilon_{\rm eff}^{(2)} + p_{\rm eff}^{(2)}\right) \;=\; 0 \quad . \tag{16e}$$

Remarks:

These alternative representations reduce the solution of the averaging problem for scalars, at least formally, to the problem of finding an 'effective equation of state' that relates the effective densities. Relativistic Lagrangian perturbation schemes will be useful to establish such relations. Looking at Eqs. (15) it is interesting to note that the 'kinematical backreaction' term itself effectively performs like a free scalar field source, or like 'stiff matter' in the case $\langle\tilde{\mathcal R}\rangle_{\mathcal D}\propto a_{\mathcal D}^{-2}$. However, care must be taken with such statements, since $\tilde{\mathcal Q}_{\mathcal D}$ is related to $\langle\tilde{\mathcal R}\rangle_{\mathcal D}$, and there is no a priori reason why $\langle\tilde{\mathcal R}\rangle_{\mathcal D}\propto a_{\mathcal D}^{-2}$ if backreaction is present. It nevertheless suggests to separate off the 'stiff component' from the 'effective equation of state': we already noted that $-\frac{1}{16\pi G}\tilde{\mathcal Q}_{\mathcal D}$ in *Corollary 1* forms a 'stiff' part; deviations from 'stiffness' are, apart from those due to the matter sources, due to the 'dynamical backreaction' $-\frac{1}{12\pi G}\tilde{\mathcal P}_{\mathcal D}$. In the form of *Corollary 2* the 'stiff' part is $-\frac{1}{16\pi G}(\tilde{\mathcal Q}_{\mathcal D} + \langle\tilde{\mathcal R}\rangle_{\mathcal D})$, and the deviations from 'stiffness' are due to the term $\frac{1}{12\pi G}(\langle\tilde{\mathcal R}\rangle_{\mathcal D} - \tilde{\mathcal P}_{\mathcal D})$. If 'dynamical backreaction' compensates the averaged scalar curvature, then the whole 'backreaction' forms a 'stiff component'. The condition for this can be inferred from Eqs. (13a,b) yielding the general relation:

$$\langle\tilde{\mathcal R}\rangle_{\mathcal D} - \tilde{\mathcal P}_{\mathcal D} = -3(\frac{\partial_t^2 a_{\mathcal D}}{a_{\mathcal D}} + 2\tilde H_{\mathcal D}^2) + 12\pi G\langle\tilde\varepsilon - \tilde p\rangle_{\mathcal D} \quad . \tag{17}$$

In general, the system of equations (13,14) is *not* a closed system, which can be most easily seen in the form of *Corollary 2*: we have three equations (16c,d,e) for the three variables $a_{\mathcal D}$, $\varepsilon_{\rm eff}^{(2)}$ and $p_{\rm eff}^{(2)}$, but only two of them are independent. We need an effective equation of state to close the system.

4. Discussion of Subcases

Since this paper is meant to provide the basic architecture for applications, let us note the following useful formulas.

Firstly, the equations of Sect. 3 simplify by using the following reparametrization of time: the line element is invariant under the change of the time coordinate $t \mapsto S(t)$, so there is still some gauge freedom. Using the 'phase fronts' as the new time coordinate we define a new lapse function $\tilde{N}$ by

$$N dt =: \tilde{N} dS \quad , \quad \text{i.e.} \quad , \quad N = \tilde{N} \partial_t S \quad . \tag{18a}$$

In particular, the total (Lagrangian) derivative becomes

$$\frac{d}{d\tau} = \frac{1}{N}\partial_t = \frac{1}{\tilde{N}}\partial_S = h\partial_S \quad , \tag{18b}$$

where the latter equality follows from Eqs. (1c) and (4b).

Notice that with this new choice of time coordinate all equations of Sect. 3 remain form invariant, if N is replaced by $\tilde{N}$, and partial time–derivatives are replaced by partial derivatives with respect to S. The latter will be abbreviated by a prime in what follows. All fields are functions of the independent variables (X^i, S) now.

Secondly, for the evaluation of the terms appearing in the averaged equations it is helpful to note the following simplifications. We shall give expressions for a barotropic fluid and, especially, for the simple class of matter models $p = \gamma\varepsilon$, which is relevant for many applications.

For barotropic fluids,

$$\dot{\varepsilon} + \theta(\varepsilon + \alpha(\varepsilon)) = 0 \quad , \tag{18c}$$

we can integrate the energy conservation law along trajectories of fluid elements using $\dot{J} = \theta J$ (Eq. 12c) and $\dot{\varrho} = -\theta\varrho$ (Eq. 9e) to find the entropy density,

$$s(\varepsilon) \propto J^{-1} \quad \text{with} \quad s(\varepsilon) \propto \exp\int \frac{d\varepsilon}{\varepsilon + \alpha(\varepsilon)} \quad , \tag{18d}$$

and, upon performing the integral, the energy density. In particular, for $\alpha' = \gamma = const.$ we obtain:

$$s(\varepsilon) \propto \varepsilon^{\frac{1}{1+\gamma}} \quad ; \quad \varepsilon \propto \frac{1}{J^{1+\gamma}} \quad . \tag{18e}$$

Hence, with $\varrho \propto J^{-1}$, and using Eq. (11g),

$$\varepsilon \propto \varrho^{1+\gamma} \quad ; \quad h \propto \varrho^{\gamma} \quad ; \quad s \propto \varrho \quad . \tag{18f}$$

Furthermore, the scaled (t–)variables are now normalized with respect to the magnitude h or its square, respectively. E.g., we have

$$\tilde{N} = \frac{1}{h} \quad ; \quad \tilde{\theta} = \tilde{N}\theta = \frac{\theta}{h} \quad ; \quad \tilde{\varepsilon} = \frac{\varepsilon}{h^2} \quad ; \quad \text{etc.} \quad . \tag{18g}$$

For barotropic fluids with $\alpha' = \gamma = const.$ the normalization function can be written in powers of the restmass density[3]. Also for this case the expression involving the covariant derivative of the lapse function in Eq. (13d) is simply proportional to the $t-$expansion rate (upon using the integral for h Eq. (18f)):

$$\dot{\tilde{N}} = \frac{\tilde{N}'}{\tilde{N}} = -\frac{h'}{h} = -\gamma\frac{\varrho'}{\varrho} = \gamma\tilde{\theta} \quad . \tag{18h}$$

The term involving the change of the lapse function in Eq. (14d) is simply time–dependent in two cases: it can be written as follows for the homogeneous case (Eq. (18i); Subsect. 4.1), and for the 'stiff fluid' representing a free minimally coupled scalar field (Eq. (18j); Subsect. 4.4):

$$\langle 2\dot{\tilde{N}}\tilde{\varepsilon}\rangle_{\mathcal{D}} = 6\tilde{H}\gamma\frac{\varepsilon_H}{h_H^2} \quad , \tag{18i}$$

$$\langle 2\tilde{\theta}\tilde{\varepsilon}\rangle_{\mathcal{D}} = 3\tilde{H}_{\mathcal{D}} \quad . \tag{18j}$$

It is interesting that the latter (fully inhomogeneous) case implies simplifications such that, e.g., Eqs. (14c,d) reduce to identities (since in this case $\tilde{\varepsilon} = \tilde{p} = \frac{1}{2}$, compare Subsect. 4.3).

For the 'dynamical backreaction' we have for $p = \gamma\varepsilon$:

$$\tilde{\mathcal{P}}_{\mathcal{D}} = \langle\tilde{\mathcal{A}}\rangle_{\mathcal{D}} + \gamma\langle\tilde{\theta}^2\rangle_{\mathcal{D}} \quad , \tag{18k}$$

and, noting that $\tilde{\theta} = -\frac{\varrho'}{\varrho}$, and using Eqs. (10a) and (18f) with $h = C_1\varrho^\gamma$, the 'dynamical backreaction' term can be entirely written in terms of the restmass density:

$$\tilde{\mathcal{P}}_{\mathcal{D}} = \langle\tilde{\mathcal{A}}\rangle_{\mathcal{D}} + \gamma\langle\tilde{\theta}^2\rangle_{\mathcal{D}} = \frac{\gamma}{C_1^2}\langle -\frac{1}{\varrho^{2\gamma+1}}\varrho^{|i}{}_{||i} + \frac{(1+\gamma)}{\varrho^{2\gamma+2}}\varrho^{|i}\varrho_{|i} + C_1^2\frac{\varrho'^2}{\varrho^2}\rangle_{\mathcal{D}} \quad . \tag{18l}$$

The same is true for the averaged spatial $t-$Ricci curvature using Eqs. (10c) and (18f) with $\varepsilon = C_2\varrho^{1+\gamma}$:

$$\langle\tilde{\mathcal{R}}\rangle_{\mathcal{D}} = 12\pi G\frac{C_2}{C_1^2}(1-\gamma)\langle\varrho^{1-\gamma}\rangle_{\mathcal{D}} - \varrho\left(\frac{1}{\varrho}\right)'' + \tilde{\mathcal{P}}_{\mathcal{D}} \quad , \tag{18m}$$

where from Eq. (11e) $C_1 = (1+\gamma)C_2$, and C_2 is determined by initial conditions. This makes the problem accessible to relativistic Lagrangian perturbation models, since the restmass density can be integrated exactly along the flow lines, $\varrho \propto J^{-1}$, and J can be computed from the basic dynamical variable in Lagrangian perturbation theory (see, e.g., Kasai 1995, Takada & Futamase 1999 for relativistic 'dust' models, Adler & Buchert 1999 for pressure–supported fluids in Newtonian theory).

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[3]The constants C_1 and C_2 appearing in the following equations are in general X^i–dependent; also equations of state, if they arise as integrals of dynamical equations, involve X^i–dependent functions of integration such as γ. One may use the freedom to relabel the fluid elements such that the constants or some product of them are X^i–independent. We here assume that all the constants are equal for each fluid element for notational ease; it is straightforward to write down the more general expressions, if needed.

4.1. Homogeneous–isotropic Cosmologies

The requirement of homogeneity and isotropy reduces Eqs. (13a,b) to the familiar Friedmann equations. We may simply put the lapse function (without loss of generality) equal to 1 in Eqs. (13a,b) and find $\tilde{\mathcal{Q}}_{\mathcal{D}} = \tilde{\mathcal{P}}_{\mathcal{D}} = 0$, and:

$$3\frac{\ddot{a}}{a} + 4\pi G(\varepsilon_H + 3p_H) \;=\; 0 \quad ; \tag{19a}$$

$$6H^2 + \mathcal{R}_H - 16\pi G\varepsilon_H \;=\; 0 \quad ; \quad H := \frac{\dot{a}}{a} \quad . \tag{19b}$$

Already before averaging ε_H and p_H are functions of time only, and the scale factor assumes its global standard value $a_{\mathcal{D}} \equiv a$. The domain dependence has disappeared. Note also that the integrability condition (14a) reduces to the equation for the 3–Ricci curvature scalar of the spatial hypersurfaces,

$$\dot{\mathcal{R}}_H + 2H\mathcal{R}_H \;=\; 0 \quad \Rightarrow \quad \mathcal{R}_H = \frac{\mathcal{R}_H^0}{a^2} \quad , \tag{19c}$$

and the averaged conservation law (14b) coincides with the local one,

$$\dot{\varepsilon}_H + 3H(\varepsilon_H + p_H) \;=\; 0 \quad . \tag{19d}$$

For a given relation between p_H and ε_H the system of equations (19) is closed.

Alternatively, we may look at the homogeneous–isotropic models within the present framework in terms of a time–dependent lapse function. For the time variable S and requiring $h = h_H(S)$, $\varepsilon = \varepsilon_H(S)$, $p = p_H(S)$, $\mathcal{R} = \mathcal{R}_H(S)$, the domain dependence disappears, and we have $\tilde{N} = h_H^{-1}$, $\tilde{\theta} = \theta_H(S)h_H^{-2}$, $\tilde{\theta}_H = 3\tilde{H}$. Setting $\sigma = 0$ we also have $\tilde{\mathcal{Q}}_{\mathcal{D}} = 0$, but $\tilde{\mathcal{P}}_{\mathcal{D}} = -\frac{h_H'}{h_H}\tilde{\theta}_H \neq 0$. The system of averaged equations Eqs. (13) together with Eqs. (14) reduces to the following set:

$$3\frac{a''}{a} + 4\pi G\frac{1}{h_H^2}(\varepsilon_H + 3p_H) = -3\tilde{H}\frac{h_H'}{h_H} \quad , \tag{20a}$$

$$6\frac{a'^2}{a^2} + \frac{\mathcal{R}_H}{h_H^2} - 16\pi G\frac{\varepsilon_H}{h_H^2} = 0 \quad , \tag{20b}$$

$$\left(\frac{\mathcal{R}_H}{h_H^2}\right)' + 2\tilde{H}\frac{\mathcal{R}_H}{h_H^2} = -2\frac{h_H'}{h_H}\left(16\pi G\frac{\varepsilon_H}{h_H^2} - 6\tilde{H}^2\right) \quad , \tag{20c}$$

$$\varrho_H' + 3\tilde{H}\varrho_H = 0 \quad , \tag{20d}$$

which, together with $h_H = \frac{\varepsilon_H + p_H}{\varrho_H}$ and an equation of state $p_H = \alpha(\varepsilon_H)$, are four equations for the four unknown functions $a(S)$, $\varrho_H(S)$, $\varepsilon_H(S)$, and $\mathcal{R}_H(S)$, but only three equations are independent. We have to use the fact that, with $p_H = \alpha(\varepsilon_H)$, ε_H can be expressed in terms of ϱ_H, which closes the system.

This set of equations looks odd with respect to the hypersurfaces $S = const$. However, upon reinvoking the covariant time derivative, e.g., $\dot{a} = h_H a'$, $\tilde{H} = \frac{\dot{a}}{a} h_H^{-1}$, h_H in the first three equations disappears, and we recover the familiar form of Eqs. (19), illustrating the covariance of the averaged equations.

4.2. Inhomogeneous 'Dust' Cosmologies

Putting in Eqs. (13) all pressure terms to zero and noticing that ε reduces to the restmass density we have $h \equiv 1$, the lapse function $\tilde{N} \equiv 1$, and the covariant time–derivative $\frac{d}{d\tau} = \partial_S$. Hence, we directly recover the form of the averaged equations of Paper I for cosmologies with a 'dust' matter content:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\langle\varrho\rangle_{\mathcal{D}} \;=\; \mathcal{Q}_{\mathcal{D}} \quad ; \tag{21a}$$

$$6H_{\mathcal{D}}^2 + \langle\mathcal{R}\rangle_{\mathcal{D}} - 16\pi G\langle\varrho\rangle_{\mathcal{D}} \;=\; -\mathcal{Q}_{\mathcal{D}} \quad ; \quad H := \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \quad , \tag{21b}$$

with the integrability condition Eq. (14a) being

$$\mathcal{Q}_{\mathcal{D}}^{\bullet} + 6H_{\mathcal{D}}\mathcal{Q}_{\mathcal{D}} + \langle\mathcal{R}\rangle_{\mathcal{D}}^{\bullet} + 2H_{\mathcal{D}}\langle\mathcal{R}\rangle_{\mathcal{D}} \;=\; 16\pi G(\langle\varrho\rangle_{\mathcal{D}}^{\bullet} + 3H_{\mathcal{D}}\langle\varrho\rangle_{\mathcal{D}}) \quad ; \tag{21c}$$

the balance equation (14d) reduces to the continuity equation for the averaged restmass density:

$$\langle\varrho\rangle_{\mathcal{D}}^{\bullet} + 3H_{\mathcal{D}}\langle\varrho\rangle_{\mathcal{D}} \;=\; 0 \quad . \tag{21d}$$

Notice that only in the case of an inhomogeneous 'dust' model we can put the lapse function $\tilde{N}$ or N itself equal to 1 without loss of generality; the averaged equations are already covariant in the comoving and synchronous gauges. It is to be emphasized that the 'dust matter model' cannot generically be foliated into hypersurfaces $S = const.$ with an inhomogeneous lapse, for Eqs. (9) necessarily imply a constant lapse function for the geodesic condition of vanishing acceleration that itself is implied by vanishing pressure.

Eqs. (21) form a set of four equations for the four unknown functions $a_{\mathcal{D}}$, $\langle\varrho\rangle_{\mathcal{D}}$, $\langle\mathcal{R}\rangle_{\mathcal{D}}$, and $Q_{\mathcal{D}}$, but only three equations are independent. As discussed in Paper I, this system cannot be closed on the level of ordinary differential equations unless additional (e.g. topological) constraints are imposed. Of course, this remark also applies to the more general matter models.

From Eqs. (21) it is obvious that the requirement $\mathcal{Q}_{\mathcal{D}} = 0$ is necessary *and* sufficient in order that $a_{\mathcal{D}}(t)$ obeys the equations of standard FLRW cosmologies.

4.3. Radiation–dominated Inhomogeneous Cosmologies

Let us consider a situation in which radiation is directly coupled to the matter fluid (according to the conjecture of a local thermodynamic equilibrium state of radiation and matter) , then we may describe the radiation cosmos as a single component perfect fluid with radiation pressure p_γ and radiation energy density ε_γ obeying $\varepsilon_\gamma = 3p_\gamma$ (Ellis 1971, see, however, Ehlers 1971). We infer already from the averaged conservation law (14b) that the time evolution of a radiation–dominated inhomogeneous universe is different from that expected from the corresponding homogeneous–isotropic world model:

$$\partial_t \langle \varepsilon_\gamma \rangle_{\mathcal{D}} + 4\tilde{H}_{\mathcal{D}} \langle \varepsilon_\gamma \rangle_{\mathcal{D}} = \langle \partial_t p_\gamma \rangle_{\mathcal{D}} - \partial_t \langle p_\gamma \rangle_{\mathcal{D}} \quad .$$

The term on the r.–h.–s. of this equation, which vanishes in the standard model, may be interpreted as an accumulated effect from inhomogeneities in the radiation field yielding deviations from a global 'equilibrium equation of state'. It should be stressed that these deviations also occur in the case where the 'backreaction' terms $\mathcal{Q}_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{D}}$ are both found or assumed to be negligible. Therefore, radiation–dominated fluids deserve further detailed study.

4.4. Inhomogeneous Scalar Field Cosmologies

Following Bruni et al. (1992) we may describe the dynamics of a scalar field ϕ, minimally coupled to gravity, in terms of the natural slicing of spacetime into a foliation of $\phi = const.$ hypersurfaces. Einstein's equations for a scalar field source are (under conditions stated below) equivalent to the phenomenological $3+1$–description of an evolving pressure–supported perfect fluid with energy–momentum tensor and corresponding fluid 4–velocity (normal to the hypersurfaces of constant ϕ) (Taub 1973, Madsen 1988):

$$T^{\phi}_{\mu\nu} = \varepsilon_\phi u_\mu u_\nu + p_\phi h_{\mu\nu} \quad ; \quad u^\mu = \frac{-\partial^\mu \phi}{\psi} \quad . \tag{22a}$$

The magnitude ψ normalizes the momentum density vector $\partial^\mu \phi$ so that $u^\mu u_\mu = -1$,

$$0 \ < \ \psi = \sqrt{-\partial^\alpha \phi \partial_\alpha \phi} = u^\mu \partial_\mu \phi =: \dot{\phi} \quad , \tag{22b}$$

where the overdot stands for the material (or Lagrangian) derivative operator as before. From Eq. (22a) we conclude that Einstein's equations feature the perfect fluid energy–momentum tensor (e.g., Madsen 1988):

$$T^{\phi}_{\mu\nu} = (\varepsilon_\phi + p_\phi) u_\mu u_\nu + p_\phi g_{\mu\nu} \ = \ \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi + V_{\rm eff}(\phi) \right) \quad , \tag{23a}$$

with

$$\varepsilon_\phi = \frac{1}{2}\psi^2 + V_{\rm eff}(\phi) \quad , \tag{23b}$$

$$p_\phi = \frac{1}{2}\psi^2 - V_{\rm eff}(\phi) \quad . \tag{23c}$$

The fluid analogy is valid, if the fluid 4–velocity is time–like and, hence, the hypersurfaces $\phi = const.$ space–like. For this to be true the 4–gradient of the scalar field has to be time–like,

$$-\psi^2 = \partial_\alpha\phi\partial^\alpha\phi < 0 \quad . \tag{23d}$$

This is a sufficient requirement for having the energy condition

$$\varepsilon_\phi + p_\phi = \psi^2 > 0 \quad , \tag{23e}$$

which follows from Eqs. (23b,c). This condition still allows for powerlaw inflation; exponential inflation is excluded and has to be studied as a separate case. This case can be studied within the fluid analogy, if we model the constant effective potential with a cosmological constant. (The basic equations have to be used including the cosmological constant – see Appendix).

We finally note that, only for constant effective potential ($V'_{\rm eff} = 0$), the restmass conservation law of a perfect fluid corresponds to the Klein–Gordon equation

$$\dot\psi + \theta\psi + V'_{\rm eff}(\phi) = 0 \quad , \quad \psi = \dot\phi \quad . \tag{23f}$$

The 'equation of state' of a scalar field is, in general, not barotropic (see: Bruni et al. 1992). However, for interesting cases it is barotropic and can be represented in terms of an 'equation of state' $p_\phi = \alpha_\phi(\varepsilon_\phi)$: 1) $\alpha_\phi = -1$ for a stationary state (vacuum ground state), and 2) $\alpha_\phi = +1$ for the free state (corresponding to a 'stiff fluid'). In general, if there exists an equation of state, it will have the form $p_\phi = \alpha_\phi(\varepsilon_\phi, s_\phi)$ with the entropy density s_ϕ. However, an evolving scalar field will in general yield a dependence of p_ϕ on the other dynamical variables $g_{\mu\nu}$ and ϕ. The function α_ϕ is determined by the dynamics and it may or may not be a priori written, e.g., as a function of the density and the entropy density.

Since the minimally coupled free scalar field (dilaton) is singled out in the present investigation as the only inhomogeneous case in which the averaged equations attain a simpler form, it is certainly worth studying this case in more detail. This is the headline of a forthcoming work (Buchert & Veneziano 2001).

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Appendix:
Basic Equations in the ADM Formalism

Let n_μ be the future directed unit normal to the hypersurface Σ. The projector into Σ, $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, ($\Rightarrow h_{\mu\nu} n^\mu = 0$, $h^\mu{}_\nu h^\nu{}_\gamma = h^\mu{}_\gamma$), induces in Σ the 3–metric

$$h_{ij} := g_{\mu\nu} h^\mu{}_i h^\nu{}_j \quad . \tag{A1a}$$

Let us write

$$n_\mu = N(-1,0,0,0) \quad , \quad n^\mu = \frac{1}{N}(1, -N^i) \quad , \tag{A1b}$$

with the lapse function N and the shift vector N^i. Note that N and N^i can be determined by the choice of coordinates.
From $n_\mu = g_{\mu\nu} n^\nu$ we find $g_{00} = -(N^2 - N_i N^i)$; $g_{0i} = N_i$; $g_{ij} = h_{ij}$ and, setting $x^0 = t$, the line element becomes:

$$ds^2 = -(N^2 - N_i N^i) dt^2 + 2 N_i dt dx^i + g_{ij} dx^i dx^j = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad . \tag{A1c}$$

Introducing the extrinsic curvature tensor on Σ by

$$K_{ij} := -n_{\mu;\nu} h^\mu{}_i h^\nu{}_j = -n_{i;j} \quad , \tag{A1d}$$

we obtain the ADM equations (Arnowitt et al. 1962, York 1979):

Energy (Hamiltonian) constraint:

$$\mathcal{R} - K^i{}_j K^j{}_i + K^2 = 16\pi G \varepsilon + 2\Lambda \quad , \quad \varepsilon := T_{\mu\nu} n^\mu n^\nu \quad ; \tag{A2a}$$

Momentum constraints:

$$K^i{}_{j||i} - K_{||j} = 8\pi G J_j \quad , \quad J_i := -T_{\mu\nu} n^\mu h^\nu{}_i \quad ; \tag{A2b}$$

Evolution equation for the first fundamental form:

$$\frac{1}{N} \partial_t g_{ij} = -2K_{ij} + \frac{1}{N}(N_{i||j} + N_{j||i}) \quad ; \tag{A2c}$$

Evolution equation for the second fundamental form:

$$\frac{1}{N} \partial_t K^i{}_j = \mathcal{R}^i{}_j + K K^i{}_j - \delta^i{}_j \Lambda - \frac{1}{N} N^{||i}{}_{||j} + \frac{1}{N} \left(K^i{}_k N^k{}_{||j} - K^k{}_j N^i{}_{||k} + N^k K^i{}_{j||k} \right)$$

$$-8\pi G (\mathcal{S}^i{}_j + \frac{1}{2} \delta^i{}_j (\varepsilon - \mathcal{S}^k{}_k)) \quad , \tag{A2d}$$

where $\mathcal{S}_{ij} := T_{\mu\nu}h^{\mu}{}_{i}h^{\nu}{}_{j}$.

For the trace parts of (A2c) and (A2d) we have:

$$\frac{1}{N}\partial_t g = 2g(-K + \frac{1}{N}N^k{}_{||k}) \quad , \quad g := \det(g_{ij}) \quad ; \tag{A2e}$$

$$\frac{1}{N}\partial_t K = \mathcal{R} + K^2 - 4\pi G(3\varepsilon - \mathcal{S}^k{}_k) - 3\Lambda - \frac{1}{N}N^{||k}{}_{||k} + \frac{1}{N}N^k K_{||k} \quad . \tag{A2f}$$

For our purpose of averaging we have used equations that correspond to the coordinate choice of vanishing shift vector. Thus, all inhomogeneities of the fluid were put into the 3–metric and the lapse function.

Assuming the tensor $T_{\mu\nu}$ has the form $T_{\mu\nu} = \varepsilon u_\mu u_\nu + p h_{\mu\nu}$ and putting the shift vector $N^i = 0$ and also $\Lambda = 0$, we obtain the equations of the main text by defining the lapse function in such a way that $a_i = \frac{N_{||i}}{N} \equiv \frac{-p_{||i}}{\varepsilon + p} = -\frac{h_{||i}}{h}$.

Notice that with this choice the unit normal coincides with the 4–velocity and, especially, the momentum flux density in Σ vanishes. The total time–derivative operator of a tensor field $\mathcal{F}$ along integral curves of the unit normal, $\frac{d}{d\tau}\mathcal{F} := n^\nu \partial_\nu \mathcal{F} = u^\nu \partial_\nu \mathcal{F}$ becomes

$$\frac{d}{d\tau}\mathcal{F} = \frac{1}{N}\partial_t \mathcal{F} \quad , \tag{A3a}$$

since $n^\nu \mathcal{F}_{||\nu} = 0$. Note that, although the definition of proper time is $\tau := \int N dt$, the line element cannot be written in the form of the comoving gauge by measuring "time" through proper time $d\tau = N dt$, since $d\tau$ is not an exact form in the case of an inhomogeneous lapse function. The exterior derivative of the proper time will involve a non–vanishing shift vector according to the space–dependence of the lapse function. Therefore, a foliation into hypersurfaces $\tau = const.$ with simultaneously requiring $u_\alpha = -\partial_\alpha \tau$ is not possible.

For vanishing shift vector the line element reads:

$$ds^2 = -N^2 dt^2 + g_{ij} dX^i dX^j \quad . \tag{A3b}$$

The lapse function itself may be written explicitly: from $h = \dot{S}$ we have:

$$N = \frac{1}{h}\partial_t S \quad . \tag{A3c}$$

Note that, if we assume that $h > 0$, implying the energy condition $\varepsilon + p > 0$, the proper time advances only in periods when the derivative $\partial_t S > 0$ which makes a difference if we consider fluids that mimick a scalar field source.

The coordinates in Σ are written in capital letters now, because for vanishing shift vector they correspond to Lagrangian coordinates as in classical fluid mechanics. In these coordinates $\frac{d}{d\tau} = \frac{1}{N}\frac{d}{dt} = \frac{1}{N}\partial_t|_{X^i}$.

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